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SOLUTION TO A PROBLEM OF C. D. GODSIL REGARDING BIPARTITE GRAPHS WITH UNIQUE PERFECT MATCHING

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We give the solution to the following question of C. D. Godsil [2]: Among the bipartite graphs G with a unique perfect matching and such that a bipartite graph obtains when the edges of the matching are contracted, characterize those having the property that $G^+ \cong G$, where G^+ is the bipartite multigraph whose adjacency matrix, B^+, is diagonally similar to the inverse of the adjacency matrix of G put in lower-triangular form. The characterization is that G must be obtainable from a bipartite graph by adding, to each vertex, a neighbor of degree one. Our approach relies on the association of a directed graph to each pair (G, M) of a bipartite graph G and a perfect matching M of G.

0. Introduction

Let G be a bipartite graph on $2n$ vertices which has a unique perfect matching, M, i.e., M is the only possible choice of n mutually disjoint edges in G. Take a proper 2-coloring of the vertices of G, thereby partitioning the vertices into two classes, R and C, each of n pairwise non-adjacent vertices. Then the $n\times n$ adjacency matrix $A(G)$, whose rows and columns are indexed by the vertex sets R and C, respectively, can be put in lower triangular form, B, (with 1's on the diagonal), by row and column permutations ([2], Lemma 2.1 and Lemma 1 below). Clearly, B is non-singular, since $\det(B)=1$, and B^{-1} has integer entries. With the additional hypothesis that the graph G/M, obtained from G by contracting the edges in M, is bipartite, Godsil ([2], Theorem 2.1, and Lemma 2 below) showed that B^{-1} is diagonally similar to a matrix B^+ whose entries are non-negative and which dominates B (that is, $B^+(i,j)\geqq B(i,j)$, for all $1\leqq i,j\leqq n$). In turn, B^+ can be regarded as the adjacency matrix of a bipartite multigraph, G^+, in which G appears as a subgraph. In this framework, C. D. Godsil posed the following question:

Problem (Godsil, [2]) Characterize the graphs G such that G^+ is isomorphic to G.

The object of this paper is to offer a solution to this problem.

For brevity, we shall refer to graphs which admit a perfect matching as *matched graphs,* and as *uni-matched* if they have only one perfect matching.

The paper is organized as follows:

First we establish a map F from the set of pairs $\{(G, M)$: G is a matched bipartite graph, M is a perfect matching of $G\}$ to the set of digraphs. This map associates an acyclic digraph to (G, M) iff M is the only perfect matching of G. In this case, the digraph can be regarded as representing a sub-relation of a partial order, and our approach relies, in a sense, on partially ordered sets.

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Lemma 1 and Lemma 2 below have been given inductive proofs in [2]. We include alternate proofs, based on our order-theoretic/digraph approach, for two reasons: first, this point of view gives insights which facilitate the demonstration of the final result; secondly, in the interest of completeness and unity of treatment. An alternate approach, based on the Coates graph associated with the matrix B (see [3]) is also outlined.

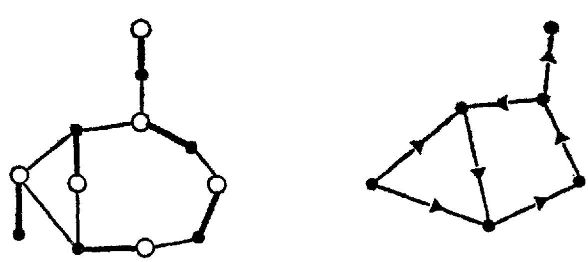
In the second section we solve Godsil's problem. The answer for the case when G is a tree is stated without proof in [2].

Results regarding (uni-)matched trees appear in [5]. The correspondence of matched trees with acyclic digraphs is examined in more detail in [6], leading to asymptotic formulae for the number of matched trees (rooted, planted, unlabeled) and for the number of self-converse directed trees.

1. Preliminary results

Let G be a matched bipartite graph on $2n$ vertices. Fix a bipartition of the vertices of G as in the introduction, into classes R and C; let M be a perfect matching of G. Then $F(G, M)$ is the directed graph $D=D(G)$ which has a vertex for each edge of M; if the vertices p, q correspond to $\{a, b\} \in M$ and $\{c, d\} \in M$, respectively, then there is a directed edge in D from p to q iff $\{a, c\}$ is an edge in G, and vertex a is in class R. Thus, the number of vertices and edges in G and D are related by $\#V(D) = \#V(G)/2$, $\#E(D) = \#E(G) - \#V(D)$. The following observation will be important in the remainder of the paper:

Lemma 0. *M is the only perfect matching of G if and only if the associated digraph $D=F(G, M)$ is acyclic.*



(G, M) $\qquad\qquad\qquad\qquad$ $D=D(G)$

Class R vertices are marked ○.

The edges in M are in heavy line.

Fig. 1. Illustration of the map $(G, M) \mapsto D=D(G)$

Proof. Note that $D=F(G, M)$ contains a directed cycle iff G contains a cycle C in which every other edge belongs to M.

Thus, if D contains a directed cycle, then, in G, M restricts to a perfect matching of a cycle C. Hence, writing C for the set of edges in this cycle, the symmetric difference $M \triangle C$ constitutes a second perfect matching of G.

Conversely, if G has two different perfect matchings, say, M_1 and M_2, then there exists a vertex v in G matched to w_1 in M_1 and to w_0 in M_2, with $w_1 \neq w_0$.

In turn, w_1 is matched in M_2 to some vertex $w_2 \neq w_1, w_0$ etc. Since G is a finite graph, this sequence of vertices must contain a cycle in which edges from M_1 and M_2 alternate. Q. e. d.

An immediate consequence of Lemma 0 is

Corollary. *Each connected component of a uni-matched bipartite graph contains at least one vertex of degree* 1 *in each color class.*

Proof. There are several possible proofs, e.g., by induction on the number of vertices, or using a lower bound on the number of matchings as a function of the minimum degree, ([4], Ch. 5). The correspondence with digraphs gives a particularly simple argument: G has a unique matching implies $D=D(G)$ is an acyclic digraph, and since it is finite, each (weak) component of D contains at least one source and one sink (vertices of in-degree and out-degree zero, respectively). These, in turn, correspond to edges in M having at least one vertex of degree 1. Q. e. d.

Let G be a uni-matched bipartite graph on $2n$ vertices, partitioned into classes R and C. The following two lemmata are prerequisites for the proof of Theorem 2.

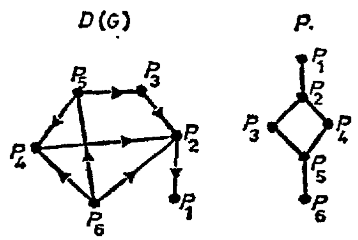
Lemma 1. ([2], Lemma 2.1.) *The $n\times n$ adjacency matrix $A(G)$ of a uni-matched bipartite graph G can be put in lower triangular form by row and column permutations.*

This can also be phrased as: any $0-1$ matrix A such that $\operatorname{per}(A)=1$ can be put in lower triangular form via row and column permutations.

Proof. Construct the acyclic digraph $D(G)$ corresponding to G and its unique matching M, as in Lemma 0. Since $D(G)$ is acyclic, it represents an antisymmetric relation on n elements to which we will refer as $\operatorname{Rel}(D)$. Let P be the partial order obtained by taking the transitive closure of $\operatorname{Rel}(D)$. Now consider any linear extension (topological sorting), L, of the poset P. Thus, L is an order preserving bijection $L\colon P\to\{1, 2, \ldots, n\}$. If $L(p)=j$, $p\in P$, then label the vertices of G so that the edge $\{v_{n+1-j}, w_{n+1-j}\}\in M$ contracts to p and $v_{n+1-j}\in R$. Finally, permute

$A(G)$

$$\begin{bmatrix} 0&0&1&0&1&0\\ 1&1&0&1&1&0\\ 1&1&0&0&0&1\\ 1&0&0&0&1&0\\ 0&0&0&0&1&1\\ 0&0&1&0&0&0 \end{bmatrix}$$



B

$$\begin{bmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 0&1&1&0&0&0\\ 0&1&0&1&0&0\\ 0&0&1&1&1&0\\ 0&1&0&1&1&1 \end{bmatrix}$$

Fig. 2. Proof of Lemma 1

the rows and columns of the adjacency matrix of G so that the i^{th} row and column are indexed by v_i and w_i, respectively. In this form, B, the $n\times n$ adjacency matrix of G is lower triangular. Q. e. d.

Lemma 2. ([2], Theorem 2.2.) *Let G be a uni-matched bipartite graph on $2n$ vertices, with perfect matching M, and with lower triangular $n\times n$ adjacency matrix B. If the* *graph G/M is bipartite then B^{-1} is diagonally similar to a matrix B^+ which dominates B.*

Proof. First, we relate our problem to the poset-theoretical Zeta and Möbius functions (see, e.g. [1], Ch. 4). Let D and P be as in the proof of Lemma 1. Let $\zeta(x)$ be the Zeta-matrix of P modified by entering the independent variable x instead of 1 for the comparable pairs of P which are not in $\mathrm{Rel}(D)$. Thus, $\zeta(0)=B$ and $\zeta(1)=\zeta_P$. Furthermore, the inverse $\mu(x)$ of the matrix $\zeta(x)$, satisfies $\mu(0)=B^{-1}$, and $\mu(1)=\mu_P$, the Möbius matrix of P. Now, it is known that $\mu_P(i,j)$ equals the value of the alternating sum of the number of chains between p_i and p_j in P, according to their length (Hall's Theorem, [1]). In matrix form,

$$\mu(x) = \sum (-1)^m[\zeta(x)-I]^m,$$

where I is the $n\times n$ identity matrix. Thus,

$$(*)\qquad \begin{aligned} B^{-1}(i,j) = \mu(0)(i,j) = \#\{&\text{even length directed paths from } p_i \text{ to } p_j \text{ in } D\}- \\ &-\#\{\text{odd length directed paths from } p_i \text{ to } p_j \text{ in } D\}. \end{aligned}$$

Since G/M is bipartite, all terms in (*) have the same sign. If $B(i,j)=1$, then there is a directed path of length 1 from p_i to p_j, hence all terms in (*) are negative and $B^{-1}(i,j)\leqq -1$. On the diagonal, $B^{-1}(i,i)=1=B(i,i)$, since D has no directed cycles, and only the zero-length path is counted in (*). Thus, $|B^{-1}(i,j)|\geqq B(i,j)$ for all $1\leqq i,j\leqq n$.

So now it will suffice to show that $|B^{-1}|$ is diagonally similar to B^{-1} where $|B^{-1}|$ is the matrix whose (i,j) entry is $|B^{-1}(i,j)|$. We will assume that G is connected; the generalization to the disconnected case is immediate. Following Godsil's idea, let S be the set of vertices in D which are joined to p_n by path(s) (not necessarily directed) of even length; since G/M i sassumed to be bipartite, S is well-defined. Consider the diagonal $(+1,-1)$-matrix where $F(i,i)=1$ iff $p_i\in S$. The claim is that $FB^{-1}F=|B^{-1}|$. Indeed, the multiplication by F on the left has changed the signs in every row indexed by a vertex in S, and then multiplication by F on the right has changed the signs in every column indexed by a vertex in S. Suppose $B^{-1}(i,j)>0$; then, by (*), there exists some even length (directed) path from p_i to p_j, and since G/M is bipartite, $i\in S$ iff $j\in S$. Consequently the sign of a positive entry in B^{-1} gets changed either 0 or 2 times when B^{-1} is multiplied on both sides by F. Similarly, a negative entry gets precisely one sign change. Thus $B^+=|B^{-1}|$.

Q. e. d.

Note. If a loop is added to each vertex of the graph D in the above discussion, the resulting graph is the Coates graph, D_B, associated with the matrix B. Thus, D_B has no directed cycles other than loops, and a well-known formula in flow graph theory (see [4]), yields (*) as the expression for the algebraic cofactor $B^*(i,j)$. Since $\det(B)=1$, $B^*(i,j)=B^{-1}(i,j)$, hence, (*) could be derived based on flow graph theory rather than partially ordered sets.

2. Solution to $G \cong G^+$

In the framework established in the previous section we can now prove

Theorem 2. *Let G be a bipartite graph on 2n vertices having a unique perfect matching, M, and such that G/M is bipartite. Then, with the notation established in the introduction, G is isomorphic to G^+ iff G can be constructed from a bipartite graph on n vertices by adding a neighbor of degree 1 to each vertex.*

Proof. First suppose that $G \cong G^+$. We begin by proving that the associated poset P has height < 2, i.e., there are no directed paths of length 2 more in D. If, on the contrary, D contained a directed path π: p_i, p_k, p_j then $B(i,j)=0$, otherwise there would be an edge between p_i and p_j, contradicting the hypothesis that G/M, which is the underlying undirected graph of D, is bipartite. On the other hand, $G \cong G^+$ is equivalent to $B=B^+$ by Lemma 2. So, in particular, $B(i,j)=0$ implies $B^+(i,j)=0$, and hence, $B^{-1}(i,j)=\mu(0)(i,j)=0$. However, we have seen that all non-zero terms in (*) have the same sign. So, each term in (*) must be zero, and this contradicts the existence of π. Now, if P has height < 2, each vertex in D has in-degree$=0$ or out-degree$=0$, and, by the construction in Lemma 0, at least one of the vertices in each matched pair in G must have degree 1. Finally, G can be constructed from the bipartite graph G/M by adding a degree 1 neighbor to each vertex.

Conversely, if G has this form, it follows immediately that P is a poset of height < 2, and Rel (D) is automatically transitively closed. Since only directed paths of length 0 and 1 exist in $P=\text{Rel}(D)$, we have $\mu_P=I-(\zeta_P-I)=-\zeta_P+2I$. Thus, $B^{-1}=-B+2I$; but then $|B^{-1}|=B$, because B has all diagonal entries equal to 1. Since, from the proof of Lemma 2, $B^+=|B^{-1}|$, we conclude that $B=B^+$.

Q. e. d.

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